WHO EATS WHOM?

Frank Harary
Research Center for Group Dynamics and Department of Mathematics,
The University of Michigan

1. Introduction

The purpose of this note is to explore some of the ramifications of applying to a zoological setting a previous interdisciplinary work involving a mathematical model for sociology. The branch of mathematics used in this model is called "graph theory" and is nonstatistical and combinatorial in nature. It provides a formula for the "status" of a person in an organization in terms of his position in the official organization chart and also for his "contrastatus." Intuitively, the contrastatus of a person is the status he has in that organization obtained from the original one by turning its organization chart upside down. We will discuss these formulas in section 3.

The zoological setting, whose validity we accept without question, is taken from Burnett, Fisher, and Zim (1958). This is a diagram containing fifteen species of animals,[1] and shows which of these species generally eat which other species. All our calculations are based on this admittedly limited list of animals. Nevertheless, it serves the purpose of illustrating a "structural classification" of animals into categories suggested by graph theory. In the last section, this classification is developed and in addition, by regarding the eating chart as an organization chart, we compute the status and contrastatus of each of the fifteen animals. We then see, subject to this criterion, which animals are relatively powerful and powerless. It is possible that this note will suggest other applications of graph theory to various relations between animals.

2. Who Eats Whom?

We begin with an alphabetical list of the fifteen animals whose eating chart is to be provided:

LIST I

1. bear	6. insect	11. salamander
2. bird	7. plants	12. skunk
3. deer	8. rabbit	13. toad
4. fox	9. raccoon	14. wildcat
5. gartersnake	10. rodent	15. wolf

The eating chart is shown in Figure 1 where each number refers to the respective animals in the above list. A directed line in this figure from

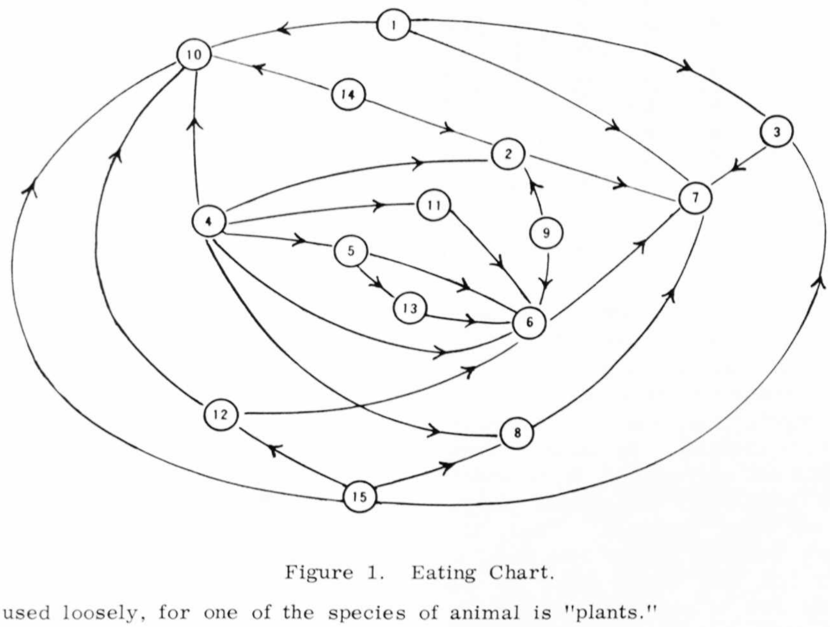

Figure 1. Eating Chart.

1. This term is used loosely, for one of the species of animal is "plants."

animal i to animal j means that the i'th eats the j'th animal. We remark briefly on the drawing of this chart. By definition, a graph is called "planar" if it can be drawn in the plane in such a way that none of its lines intersect each other. We note that in Figure 1 there occurs exactly one intersection among the lines of this directed graph. There is a well-known mathematical theorem which provides a criterion for planarity of a graph. According to this theorem, the directed graph of Figure 1 is not planar. Therefore it is not possible for anyone to draw Figure 1 in such a way that there are no intersections.

We now have the eating matrix which goes with Figure 1. Both the row index and the column index stand for the indicated animal among the list of fifteen animals, and the rule for the construction of this matrix is as follows: there is a one in the i, j, place whenever the animal i eats animal j and there is a zero in the i, j place otherwise. The row sums and column sums are shown to the right and below this matrix respectively.

EATING MATRIX

eating animal \ eaten animal	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	row sum
1	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	3
2	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
4	0	1	0	0	1	1	0	1	0	1	1	0	0	0	0	6
5	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	2
6	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
9	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	2
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	2
13	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
14	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	2
15	0	0	1	0	0	0	0	1	0	1	0	1	0	0	0	4
column sum	0	3	2	0	1	6	5	2	0	5	1	1	1	0	0	27

The i'th row sum shows the number of animals which animal i eats, while the j'th column sum is the number of animals which eat animal j.

It is possible to read from this eating matrix which animals are vegetarians, which are insectarians, and which carnivarians. By these terms we mean animals which eat only plants, those which eat only insects, and those which eat neither plants nor insects but do eat other animals. Thus, the vegetarians are those animals whose row sum in the eating matrix is one, and whose entry in the seventh column of its row is one. A similar description using the sixth column determines the insectarians. Finally, the carnivarians are those animals with a positive row sum in whose row the entries in both columns six and seven are zero. We immediately see from the eating matrix that the vegetarians are animals 2, 3, 6, and 8, i.e., bird, deer, insect, and rabbit. The insectarians are animals 11 and 13, the salamander and toad. The carnivarians are animals 14 and 15, the wildcat and the wolf.

3. Status and Contrastatus

It is most convenient to quote the definition of status from a previous article (1959).

> The status of a person A in an organization is the number of his immediate subordinates plus twice the number of their immediate subordinates (who are not immediate subordinates of A) plus three times the number of their immediate subordinates (not already included), etc.

We illustrate this formula with a standard organization chart shown in Figure 2.

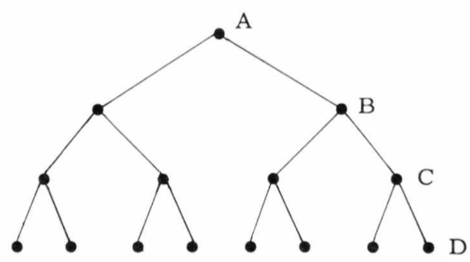

Figure 2. Standard Organization Chart.

Let s(A) denote the status of A. Then in Figure 2, we have the following values of the status of the four members shown by letters:

$$s(A) = 34$$
$$s(B) = 10$$
$$s(C) = 2$$
$$s(D) = 0$$

Let s'(A) denote the contrastatus of A, that is, the status of A in the organization chart obtained by reversing the direction of every directed line in the chart. Then in Figure 2, we have the following values of the contrastatus of the same four individuals:

$$s'(A) = 0$$
$$s'(B) = 1$$
$$s'(C) = 3$$
$$s'(D) = 7$$

The net status of a person is obtained by subtracting his contrastatus from his status. In the next section we will calculate the status, contrastatus, and net status for each of the fifteen animals.

4. Powerful and Powerless Animals

In the following list, s is the status of an animal, s' is his contrastatus, and $\Delta s = s - s'$, his net status.

LIST II

animal	s	s'	Δs
1	3	0	3
2	1	3	-2
3	1	2	-1
4	10	0	10
5	4	1	3
6	1	8	-7
7	0	22	-22
8	1	2	-1
9	4	0	4
10	0	4	-4
11	3	1	2
12	4	1	3
13	3	3	0
14	4	0	4
15	8	0	8

It is interesting to list the animals with the net status of each in the order of decreasing net status. This is done in the following list.

Within this animal population we thus find that the fox has the highest net status, with the wolf taking second place. On the other end of the scale, the plants occupy the lowest position by far, while the three next lowest places are filled by insects, rodents, and birds respectively.

LIST III

animal	Δs
4	10
15	8
9,14	4
1,5,12	3
11	2
13	0
3,8	-1
2	-2
10	-4
6	-7
7	-22

In the theory of directed graphs, a transmitter is defined as a point to which there are no directed lines and from which there is at least one directed line, while a receiver has no lines from it but there is at least one directed line to such a point. A carrier point is a point of a directed graph with exactly one line to it and exactly one line from it. Thus in terms of the eating chart of Figure 1, a transmitter is an animal which eats other animals but is not eaten, while a receiver is an animal which is eaten by others but does not eat any. A carrier is an animal which eats exactly one other animal and which is in turn eaten by exactly one other animal. These three kinds of animal can also be determined by inspection of the eating matrix. For a transmitter has positive row sum and zero column sum, a receiver has positive column sum and zero row sum, and a carrier has both row sum and column sum one. We find that the transmitters are animals 1, 4, 9, 14, and 15, the receivers are 7 and 10, and the carriers are animals 11 and 13. It is interesting to note that the carriers coincide with the insectarians and that both carnivarians are transmitters.

A directed graph represents an irreflexive relation if there are no directed lines from a point to itself. It represents an asymmetric relation if there are no symmetric pairs of directed lines, i.e., no pair of lines joining the same two points in both directions. In these terms, we find that the eating chart presents a relation which is both irreflexive and asymmetric. A relation is called transitive if whenever i is in the relation to j and j is in the relation to k, then i is in the relation to k. Thus a directed graph represents a transitive relation if and only if whenever there is a two-step directed path from one animal to another, then the one animal always eats the other. In the article by Harary and Paper (1957), a formula of the degree of transitivity of a binary relation was proposed for the purpose of making certain linguistic measurements. It would be interesting to apply this formula to the eating chart to calculate the degree of transitivity of the eating relation.

REFERENCES

1. R. W. BURNETT, H. I. FISHER, and H. S. ZIM, Zoology, Simon and Schuster: New York, 1958; p. 146.
2. F. HARARY, Status and contrastatus, Sociometry, 1959, 22, 23-43.
3. F. HARARY, and H. H. PAPER, Toward a general calculus of phonemic distribution, Language, 1957, 33, 143-169.